Polynomial integration

Name……

(1) Integrate the following:

(a) $\int 3x^2 + 5\,dx$ (b) $\int 0.25x^4 + \sqrt{x}\,dx$ (c) $\int x\,dx$ (d) $\int dx$

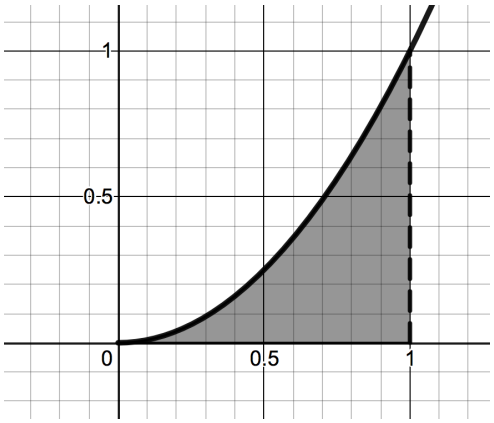
(a) $x^3 + 5x + c$ (b) $0.05x^5 + \frac{2x^{1.5}}{3} + c$ (c) $\frac{x^2}{2} + c$ (d) $x + c$

(2) Integrate the following:

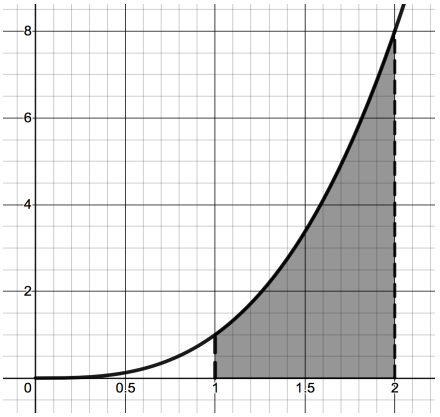
(a) $\int \frac{2}{x^2}\,dx$ (b) $\int \sqrt[3]{x}\,dx$ (c) $\int 3 + 6x + 3x^2\,dx$ (d) $\int \frac{3}{5x^3}\,dx$

(a) $-2x^{-1} + c$ (b) $\frac{3x^{\frac{4}{3}}}{4} + c$ (c) $3x + 3x^2 + x^3 + c$ (d) $-0.3x^{-2} + c$

(3) Find the shaded area for the following graph:



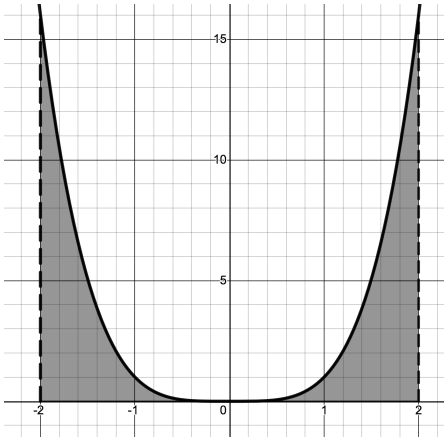
$f(x) = x^2$



$f(x) = x^3$

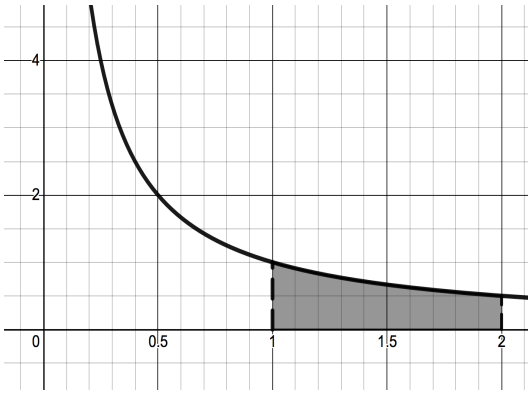
$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$\int_1^2 x^3\,dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{15}{4}$



$f(x) = x^4$

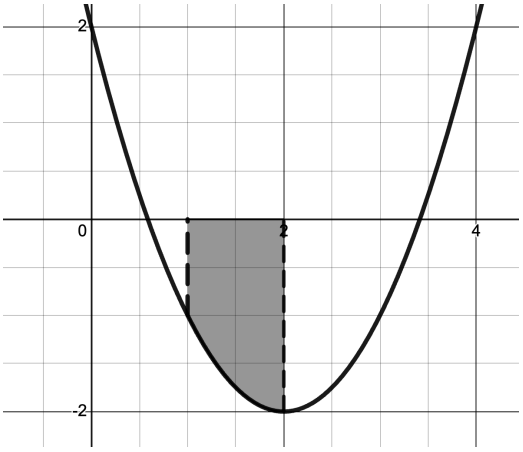
$\int_{-2}^{2} x^4\,dx = \left[\frac{x^5}{5}\right]_{-2}^{2} = \frac{64}{5}$



$f(x) = \frac{1}{x}$

$\int_{1}^{2} \frac{1}{x}\,dx = [lnx]_1^2 = ln2$

(4) The following graph $x^2 - 4x + 2$ is shaded between $x = 1$ and $x = 2$.



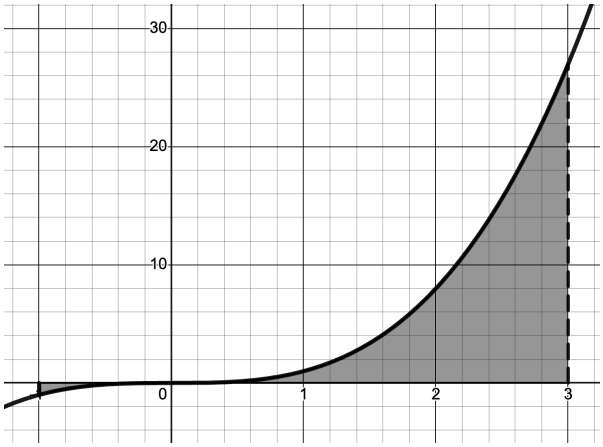
(a) Find $\int_1^2 x^2 - 4x + 2\,dx$

$$\int_1^2 x^2 - 4x + 2\,dx = \left[\frac{x^3}{3} - 2x^2 + 2x\right]_1^2 = -\frac{5}{3}$$

(b) Explain why your result in (a) is not the same as the shaded **area**.

Integrals underneath the curve are negative – area is always positive.

(5) The following graph x^3 is shaded between $x = -1$ and $x = 3$.



(a) Find $\int_{-1}^{3} x^3\, dx$

$$\int_{-1}^{3} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{3} = 20$$

(b) Find $\int_{-1}^{0} x^3\, dx$ and $\int_{0}^{3} x^3\, dx$.

$$\int_{-1}^{0} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{0} = -\frac{1}{4}$$

$$\int_{0}^{3} x^3 dx = \left[\frac{x^4}{4}\right]_{0}^{3} = 20.25$$

(c) Explain how to use your answer to part (b) to find the total shaded **area**.

$$\text{Total area} = \left|-\tfrac{1}{4}\right| + 20.25 = 20.5$$

Polynomial integration II

Name……

(1) Integrate the following:

(a) $\int_0^3 3x^2\,dx$ (b) $\int_{-2}^1 -x^2\,dx$ (c) $\int_1^2 x^3+1\,dx$ (d) $\int_1^4 \sqrt{x}\,dx$

$[x^3]_0^3 = 27$ $\left[-\frac{x^3}{3}\right]_{-2}^1 = -3$ $\left[\frac{x^4}{4}+x\right]_1^2 = \frac{19}{4}$ $\left[\frac{2x^{1.5}}{3}\right]_1^4 = \frac{14}{3}$

(2) Integrate the following:

(a) $\int_1^3 \frac{2}{x}dx$ (b) $\int_0^8 \sqrt[3]{x}\,dx$ (c) $\int_1^2 x^2-x\,dx$ (d) $\int_3^5 \frac{1}{x^2}\,dx$

$[2lnx]_1^3 = 2ln3$ $\left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^8 = 12$ $\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2 = \frac{5}{6}$ $[-x^{-1}]_3^5 = \frac{2}{15}$

(3) Integrate the following:

(a) $\int_{-1}^3 (t+2)(t+3)dt$ (b) $\int_1^2 \frac{t^2+1}{t^2}\,dt$ (c) $\int_0^1 \sqrt{\frac{t^3}{4}}\,dt$

$\int_{-1}^3 t^2+5t+6dt$ $\int_1^2 1+\frac{1}{t^2}\,dt$ $\int_0^1 \frac{t^{1.5}}{2}\,dt$

$=\frac{160}{3}$ $=\frac{3}{2}$ $=0.200$

(4) For the following integrals, find a.

(a)

$$\int_1^3 ax^2\,dx = 52$$

$$a\left[\frac{x^3}{3}\right]_1^3 = 52$$

$$a = 6$$

(b)

$$\int_2^a 4x^3\,dx = 240$$

$$[x^4]_2^a = 240$$

$$a^4 - 16 = 240$$

$$a = 4$$

(5) A curve has gradient function defined below. If the curve passes through the point (1,10), find the equation of the curve.

$$\frac{dy}{dx} = 4x + 3$$

$$y = 2x^2 + 3x + c$$

$$10 = 2(1)^2 + 3(1) + c$$

c=5

$$y = 2x^2 + 3x + 5$$

(6) The gradient of a function is defined below. If the curve passes through the point (2,0), find the equation of the curve.

$$f'(x) = 2x^2 - x - 3$$

$$f(x) = \frac{2x^3}{3} - \frac{x^2}{2} - 3x + c$$

$$f(2) = \frac{2(2)^3}{3} - \frac{(2)^2}{2} - 3(2) + c = 0$$

c=2.6666666

$$f(x) = \frac{2x^3}{3} - \frac{x^2}{2} - 3x + \frac{8}{3}$$

(7) The rate of growth of a population (P) of foxes in a park over time (t) in years is given by the equation:

$$\frac{dP}{dt} = 10\sqrt{t}$$

(a) If there were initially 100 foxes in the park find an equation for P.

$$P = \frac{20}{3}t^{1.5} + c$$

$$100 = \frac{20}{3}(0)^{1.5} + c$$

$$P = \frac{20}{3}t^{1.5} + 100$$

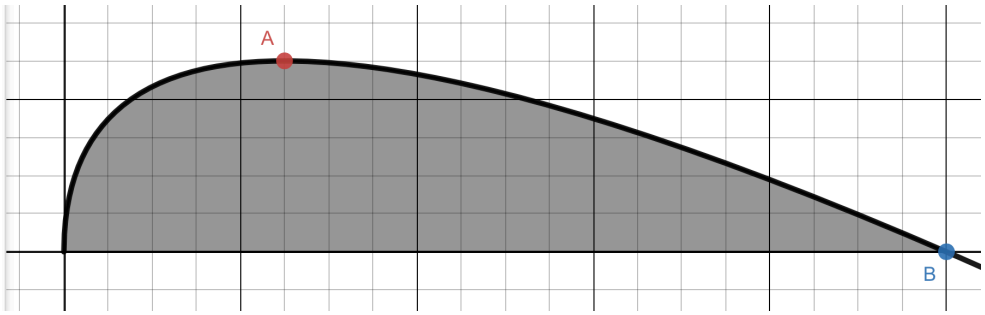
(b) After how many complete years will there be more than 200 foxes?

$$\frac{20}{3}t^{1.5} + 100 > 200$$

$$t = 6.08$$

$$t = 7$$

(8) The graph of $f(x) = \sqrt{x} - x$ is shown below:



(a) Find the coordinates of A.

$$f'(x) = 0.5x^{-0.5} - 1$$

$$0.5x^{-0.5} - 1 = 0$$

$$\frac{1}{\sqrt{x}} = 2$$

$$x = 0.25$$

$$(0.25, 0.25)$$

(b) Find the coordinates of B.

$$\sqrt{x} - x = 0$$

$$\sqrt{x}(1 - \sqrt{x}) = 0$$

$$x = 1$$

$$(1, 0)$$

(c) Find the shaded area.

$$\int_0^1 \sqrt{x} - x = \left[\frac{2x^{1.5}}{3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{6}$$

Integration rules: non polynomials

Name……

(1) Integrate the following:

(a) $\int cos\theta \, d\theta$ (b) $\int cos2\theta \, d\theta$ (c) $\int sin3\theta \, d\theta$ (d) $\int -2sin2\theta \, d\theta$

(a) $sin\theta + c$ (b) $\frac{sin2\theta}{2} + c$ (c) $-\frac{cos3\theta}{3} + c$ (d) $cos2\theta + c$

(e) $\int \sin(2\theta + 1) \, d\theta$ (f) $\int -3cos\left(\frac{\theta}{2}\right) \, d\theta$ (g) $\int \cos(4\theta - 2) \, d\theta$

(e) $-0.5\cos(2\theta + 1) + c$ (f) $-6sin\left(\frac{\theta}{2}\right) + c$ (g) $0.25\sin(4\theta - 2) + c$

(h) $\int 2\sin(\pi\theta + 2) \, d\theta$

(h) $-\frac{2}{\pi}\cos(\pi\theta + 2)$

(2) Integrate the following:

(a) $\int e^{2x} \, dx$ (b) $\int e^{5x+1} \, dx$ (c) $\int e^{0.5x} \, dx$ (d) $\int e^{-x} \, dx$

(a) $0.5e^{2x} + c$ (b) $0.2e^{5x+1} + c$ (c) $2e^{0.5x} + c$ (d) $-e^{-x} + c$

(e) $\int e^{7x+4} \, dx$

(e) $\frac{1}{7}e^{7x+4} + c$

(3) Integrate the following:

(a) $\int (2+x)^5\,dx$ (b) $\int (4+x)^7\,dx$ (c) $\int (2+2x)^3\,dx$

(a) $\frac{1}{6}(2+x)^6+c$ (b) $\frac{1}{8}(4+x)^8+c$ (c) $\frac{1}{8}(2+2x)^4+c$

(d) $\int (5x-2)^6\,dx$ (e) $\int (-x+1)^4\,dx$ (f) $\int (0.5x+2)^3\,dx$

(d) $\frac{1}{35}(5x-2)^7+c$ (e) $-\frac{1}{5}(-x+1)^5+c$ (f) $\frac{1}{2}(0.5x+2)^4+c$

(4) Integrate the following:

(a) $\int \frac{1}{x}\,dx$ (b) $\int \frac{1}{x+1}\,dx$ (c) $\int \frac{1}{x-5}\,dx$ (d) $\int \frac{1}{2x+5}\,dx$

(a) $\ln(x)+c$ (b) $\ln(x+1)+c$ (c) $\ln(x-5)+c$ (d) $0.5\ln(2x+5)+c$

(e) $\int \frac{1}{3x+4}\,dx$ (f) $\int \frac{2}{3x-1}\,dx$

(e) $\frac{1}{3}\ln(3x+4)+c$ (d) $\frac{2}{3}\ln(3x-1)+c$

(5) Integrate the following:

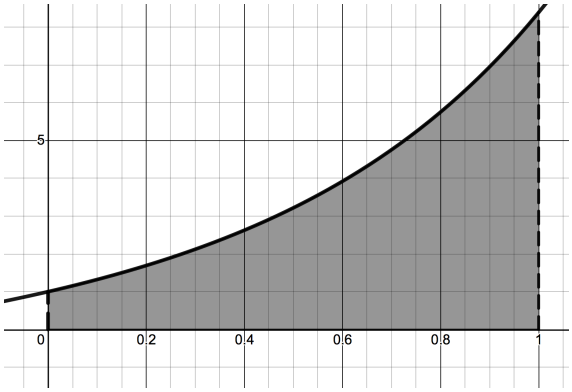
(a) $\int \frac{1}{(x+5)^3}\,dx$ (b) $\int \frac{1}{(2x+3)^2}\,dx$

(a) $-\frac{1}{2}(x+5)^{-2}+c$ (b) $-\frac{1}{2}(2x+3)^{-1}+c$

(c) $\int \sqrt{4x+1}\,dx$ (d) $\int \frac{1}{\sqrt{2x+5}}\,dx$

(c) $\frac{1}{6}(4x+1)^{1.5}+c$ (d) $(2x+5)^{0.5}+c$

(6) Below is part of the graph of $y = e^{2x} + x$

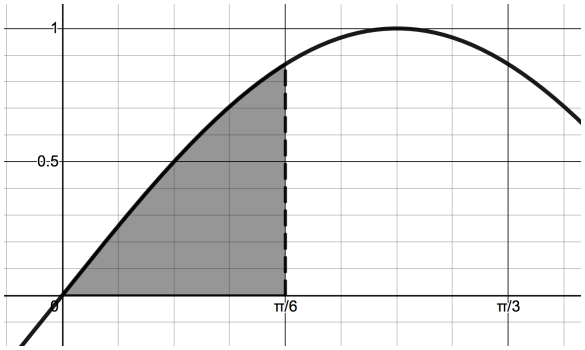


Find the area of the shaded region, leaving your answer in terms of e.

$$\int_0^1 e^{2x} + x \, dx$$

$$[0.5e^{2x} + 0.5x^2]_0^1 = 0.5e^2$$

(7) Below is part of the graph of $y = \sin(2x)$



Find the area of the shaded region.

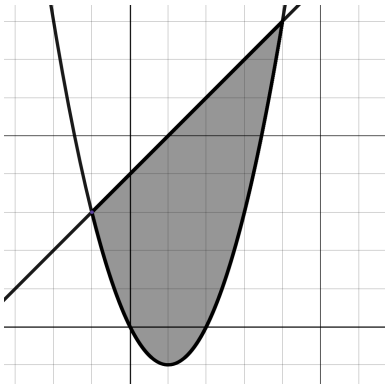
$$\int_0^{\frac{\pi}{6}} \sin(2x) \, dx$$

$$[-0.5\cos(2x)]_0^{\frac{\pi}{6}} = 0.25$$

Areas between curves and Fundamental Theorem of Calculus

Name……

(1) The graph below shows the line $y = x + 4$ and the curve $y = x^2 - 2x$



(a) Find the x-coordinates of the intersection between the line and the curve.

$$x^2 - 2x = x + 4$$

$$x^2 - 3x - 4 = 0$$

$$x = -1, \qquad x = 4$$

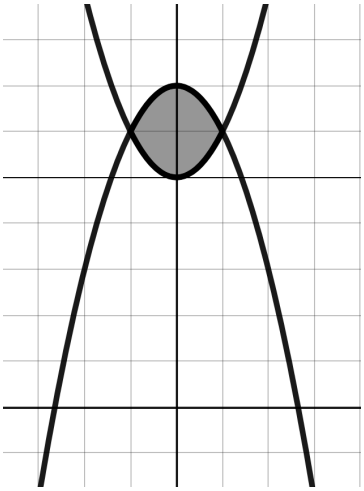
(b) Hence find the shaded area.

$$\int_{-1}^{4} x + 4 - (x^2 - 2x)dx$$

$$\int_{-1}^{4} -x^2 + 3x + 4dx = \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^{4}$$

$$= \frac{125}{6}$$

(2) The graph below shows the line $y = -x^2 + 7$ and the curve $y = x^2 + 5$



Find the shaded area.

$$-x^2 + 7 = x^2 + 5$$

$$2x^2 + 2$$

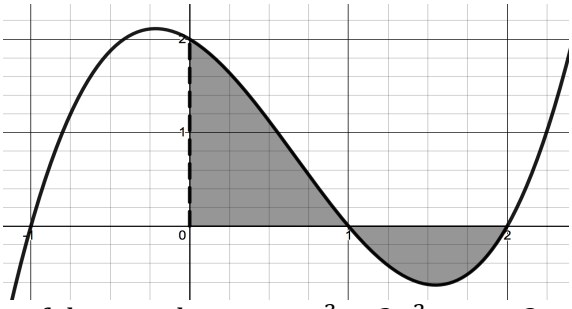
$$x = \pm 1$$

$$\int_{-1}^{1} -x^2 + 7 - (x^2 + 5)dx$$

$$\int_{-1}^{1} -2x^2 + 2dx = \left[-\frac{2x^3}{3} + 2x\right]_{-1}^{1}$$

$$= \frac{8}{3}$$

(3) A cubic graph is shown below:



(a) Show that the equation of this graph is $y = x^3 - 2x^2 - x + 2$

$$y = a(x+1)(x-1)(x-2)$$

$$y = a(x^2-1)(x-2)$$

$$y = a(x^3 - 2x^2 - x + 2)$$

When $x = 0, y = 2$. Therefore $a = 1$

$$y = x^3 - 2x^2 - x + 2$$

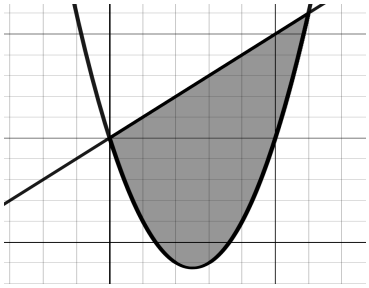
(b) Find the total shaded area.

$$\int_0^1 x^3 - 2x^2 - x + 2dx = \frac{13}{12}$$

$$\int_1^2 x^3 - 2x^2 - x + 2dx = -\frac{5}{12}$$

$$\frac{13}{12} + \left|-\frac{5}{12}\right| = \frac{3}{2}$$

(4) The graph below shows the line $y = x + 5$ and the curve $y = x^2 - 5x + 5$



Find the total shaded area

$$x^2 - 5x + 5 = x + 5$$

$$x^2 - 6x = 0$$

$$x = 0, \qquad x = 6$$

$$\int_0^6 x + 5 - (x^2 - 5x + 5)dx$$

$$\int_0^6 -x^2 + 6x\, dx = \left[-\frac{x^3}{3} + 3x^2\right]_0^6 = 36$$

(5) Given that:

$$\int_{-1}^{3} f(x)dx = 8, \quad \int_{-1}^{5} f(x)dx = 2, \quad \int_{-1}^{5} g(x)dx = 5,$$

Find:

(a) $\int_{-1}^{5} f(x) + 2g(x)\, dx$

$2 + 2(5) = 12$

(b) $\int_{3}^{5} f(x)dx$

$\int_{-1}^{5} f(x)dx - \int_{-1}^{3} f(x)dx$

$2 - 8 = -6$

(c) $\int_{-1}^{5} f(x) + 2\, dx,$

$$2 + \int_{-1}^{5} 2\, dx = 14$$

(d) $\int_{1}^{5} f(x - 2)\, dx$

$$\int_{1}^{5} f(x-2)\, dx \quad = \int_{-1}^{3} f(x)\, dx \quad = 8$$

Kinematics I

Name……

(1) The displacement of a particle travelling along a straight line is given by the equation:

$$s(t) = t^3 - 2t^2 + 4$$

(a) Find the particle's initial displacement.

$$s(0) = 4$$

(b) Find an equation for the particle's velocity, v.

$$v(t) = 3t^2 - 4t$$

(c) When is the particle at rest?

$$v(t) = 3t^2 - 4t = 0$$

$$t = 0, t = \frac{4}{3}$$

(ii) At what times is $v = -1$?

$$v(t) = 3t^2 - 4t = -1$$

$$3t^2 - 4t + 1 = 0$$

$$t = 1, t = \frac{1}{3}$$

(d) What is the acceleration of the particle when $t = 1$?

$$a(t) = 6t - 4$$

$$a(1) = 2$$

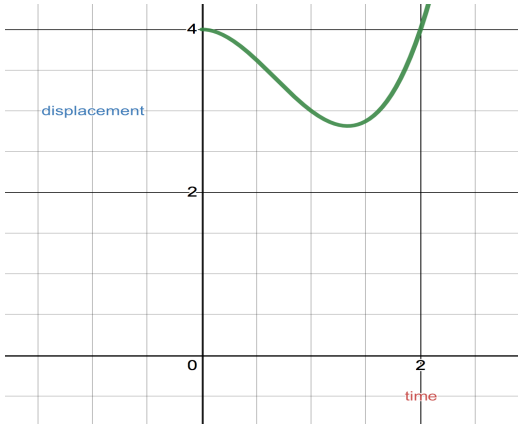
(ii) When is the particle travelling at its minimum velocity?

$$a(t) = 6t - 4 = 0$$

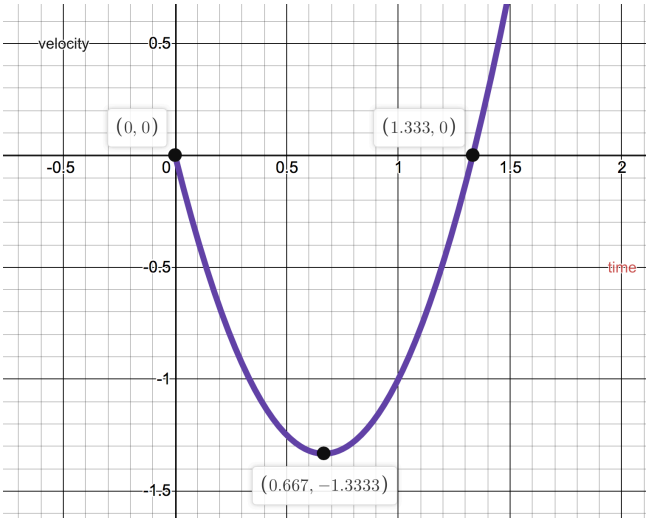
$$t = \frac{2}{3}$$

Graphs for Q1:

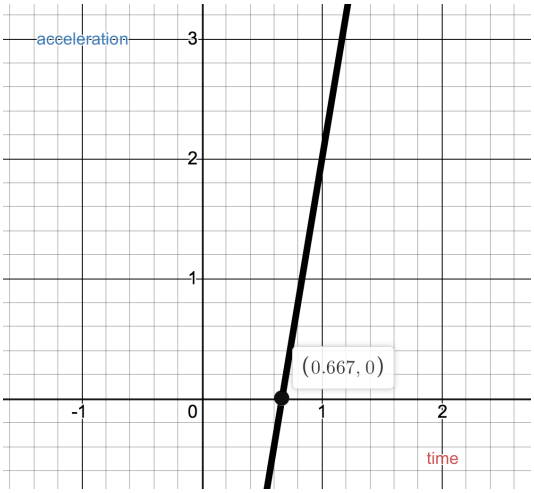
Displacement:



Velocity:



Acceleration:



(2) The displacement, from the origin O, of a particle after t seconds, travelling along a straight line is given by the equation:

$$s(t) = \sin(3t) + 2, \qquad 0 \leq t \leq 2.$$

(a) How far from O is the particle when $t = \frac{\pi}{9}$?

$$s\left(\frac{\pi}{9}\right) = \sin\left(3\frac{\pi}{9}\right) + 2 = \frac{\sqrt{3}}{2} + 2$$

(b) Explain why the particle will never be at the origin O.

If at the origin then $s(t) = 0$.

$$\sin(3t) + 2 = 0$$

$\sin(3t) = -2$ and this has no solutions

(c) Find an expression for the velocity of the particle.

$$v(t) = 3\cos(3t)$$

(d) At what times does the particle change direction in the first 1 second?

Change direction when $v(t) = 0$

$$v(t) = 3\cos(3t) = 0$$

$$\cos(3t) = 0$$

$$\cos(\theta) = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$t = \frac{\pi}{6}, \frac{\pi}{2}$$

(d) What is the maximum velocity the particle reaches?

$$-3 \leq v(t) = 3\cos(3t) \leq 3$$

maximum velocity = 3

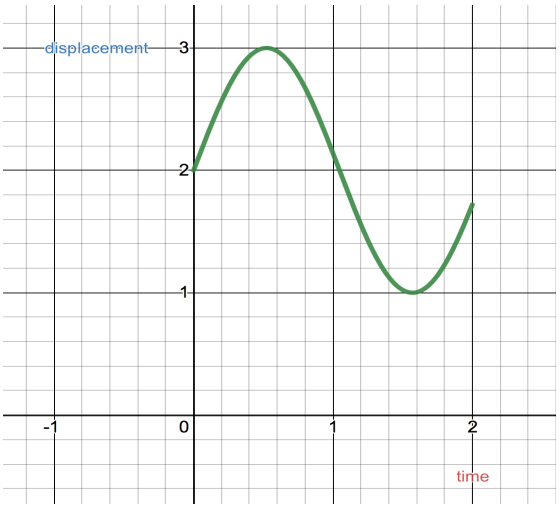
(e) What is the acceleration of the particle when $t = \frac{\pi}{9}$?

$$a(t) = -9\sin(3t)$$

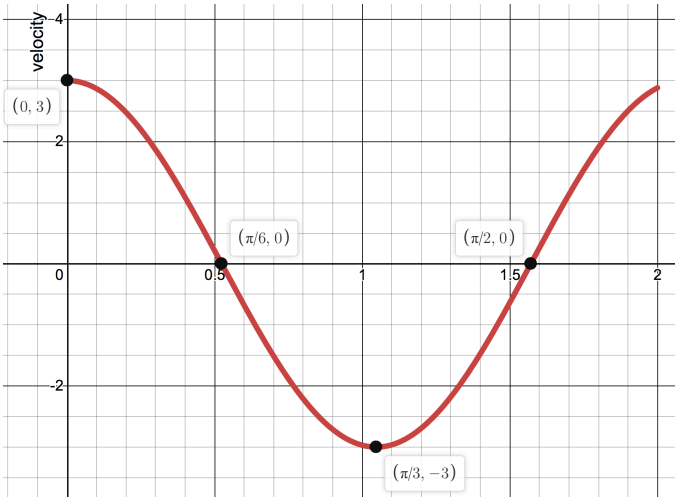
$$a\left(\frac{\pi}{9}\right) = -9\sin\left(3\frac{\pi}{9}\right) = -\frac{9\sqrt{3}}{2}$$

Graphs for Q2

Displacement:



Velocity:



(3) The velocity of a particle travelling along a straight line after t seconds is given by the equation:

$$v(t) = t^2 - 3t + 2$$

(a) Find the particle's initial velocity.

$$v(0) = 2$$

(b) Find when the particle changes direction.

$$v(t) = t^2 - 3t + 2 = 0$$

$$t = 2, t = 1$$

(ii) Find the minimum velocity.

$$a(t) = 2t - 3 = 0$$

$$t = 1.5$$

$$v(t) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 = -0.25$$

(c) Find the displacement of the particle when $t = 2$

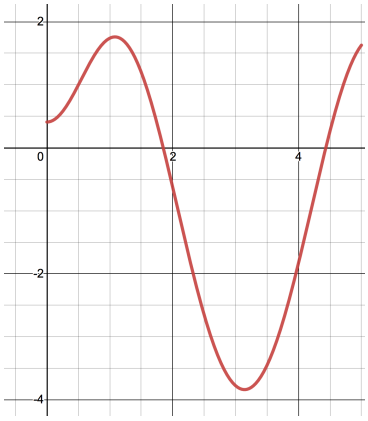
$$\int_0^2 t^2 - 3t + 2\, dx = \frac{2}{3}$$

(d) Find the total distance travelled over the first 2 seconds.

$$\left|\int_0^1 t^2 - 3t + 2\, dx\right| + \left|\int_1^2 t^2 - 3t + 2\, dx\right| = \frac{5}{6} + \frac{1}{6} = 1$$

(4) [GDC REQUIRED]. The graph below shows the velocity $v\ ms^{-1}$ of a particle P along a straight line after t seconds.

$$v(t) = 8\cos(t) - 5e^{\cos(t)} + 6, \qquad 0 \le t \le 5$$



(a) Find the initial velocity

0.409

(b) Write down the maximum **speed** of the particle.

3.84

(c) Write down the number of times the particle changes direction.

2 times

(d) Find the time when the acceleration is first 0 ms^{-2}.

$t = 1.08$

(e) Find the acceleration when $t = 1$

$$\frac{d}{dt}\left(8\cos(t) - 5e^{\cos(t)} + 6\right)_{t=1} = 0.490$$

(f) Find the total distance travelled in the first 5 seconds.

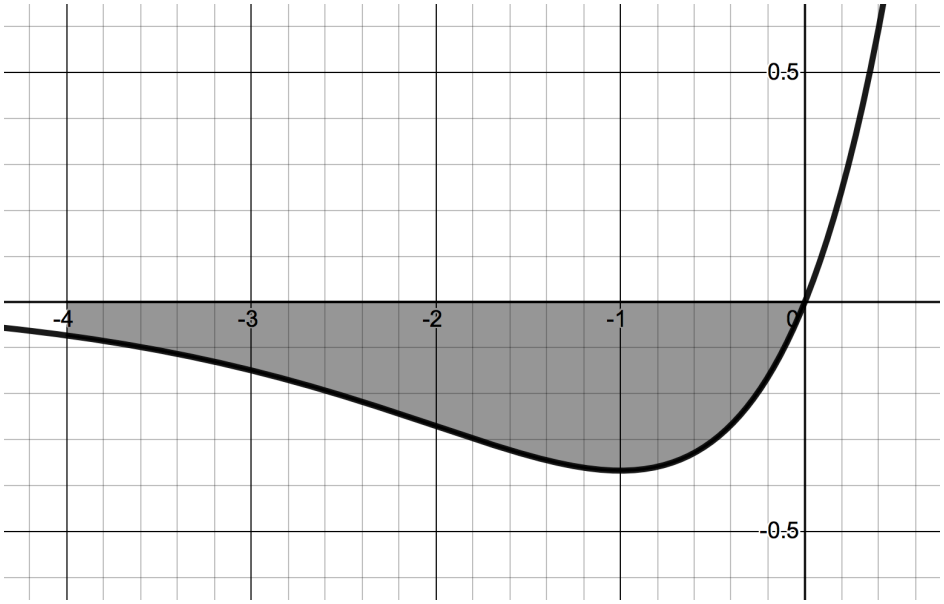
$$\int_0^5 \left|8\cos(t) - 5e^{\cos(t)} + 6\right| = 8.76$$

Integration by parts

Name……

(1) You are given the following function:

$$f(x) = xe^x$$



Part of this graph is shown above.

(a) Find $f'(x)$.

$$f(x) = xe^x$$

$$f'(x) = e^x + xe^x$$

(ii) **Hence** find the exact coordinates of the minimum point. Leave your answer in terms of e.

$$f'(x) = e^x + xe^x = 0$$

$$e^x(1 + x) = 0$$

$$x = -1$$

$$(-1, -e^{-1})$$

(b) Find the exact area of the shaded region. Leave your answer in terms of e.

$$\int xe^x dx$$

$$u = x, \qquad u' = 1, \qquad v' = e^x, \qquad v = e^x$$

$$\int uv'dx = uv - \int u'v$$

$$\int xe^x dx = xe^x - \int e^x$$

$$\int xe^x dx = xe^x - e^x + c$$

$$\int_{-4}^{0} xe^x dx = [xe^x - e^x]_{-4}^{0}$$

$$-1 - (-4e^{-4} - e^{-4})$$

$$-1 + 5e^{-4}$$

(c) Find a such that $\int_a^0 f(x)dx = -f'(a)$. Leave your answer in the form $pln(q)$ where $q \in \mathbb{Z}$.

$$-f'(a) = -e^a - ae^a$$

$$\int_a^0 f(x)dx = [xe^x - e^x]_a^0$$

$$-e^a - ae^a = [xe^x - e^x]_a^0$$

$$-e^a - ae^a = -1 - (ae^a - e^a)$$

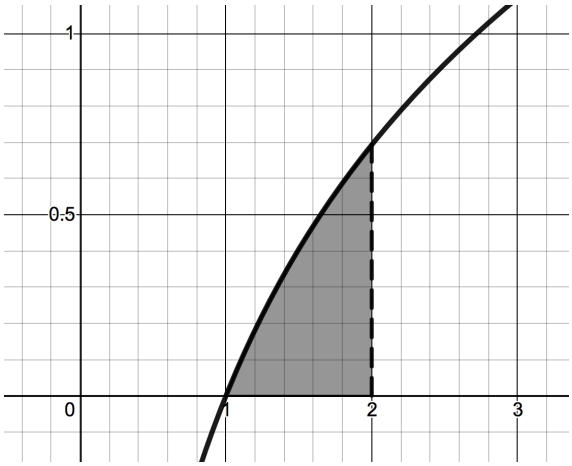
$$2e^a = 1$$

$$e^a = \frac{1}{2}$$

$$a = ln\left(\frac{1}{2}\right) = -\ln(2)$$

(2) You are given the following function:

$$f(x) = \ln(x)$$



Part of this graph is shown above.

(a) Find the shaded area. Give your answer in terms of $ln2$.

$$\int 1 \ln(x)\, dx$$

$$u = \ln(x), \qquad u' = \frac{1}{x}, \qquad v' = 1, \qquad v = x$$

$$\int uv'dx = uv - \int u'v$$

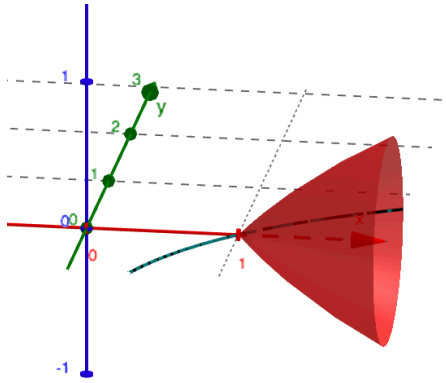
$$\int \ln(x)dx = xln(x) - \int 1dx$$

$$\int \ln(x)dx = xln(x) - x + c$$

$$\int_1^2 \ln(x)dx = [xln(x) - x]_1^2$$

$$= 2ln(2) - 1$$

(b) The area between the curve $f(x)$ and the lines $x = 1, x = 2$ is rotated 360 degrees about the x-axis.



Find the volume of the shape formed. Give your answer in terms of $ln2$.

$$\int \ln(x)\ln(x)\,dx$$

$$u = \ln(x), \qquad u' = \frac{1}{x}, \qquad v' = \ln(x), \qquad v = xln(x) - x$$

$$\int uv'dx = uv - \int u'v$$

$$\int \ln(x)\ln(x)\,dx = x(\ln(x))^2 - xln(x) - \int \ln(x) - 1dx$$

$$= x(\ln(x))^2 - xln(x) - (xln(x) - x - x) + c$$

$$= x(\ln(x))^2 - 2xln(x) + 2x + c$$

Therefore:

$$\pi \int_1^2 lnxlnx = \pi[x(\ln(x))^2 - 2xln(x) + 2x]_1^2$$

$$= \pi(2(\ln 2)^2 - 4ln2 + 2)$$

Substitution and difficult integration HL

Name……

(1) Integrate the following:

(a) $\int \frac{1}{4+x^2}\,dx$ (b) $\int \frac{1}{1+4x^2}\,dx$

(a) $\frac{1}{2}\arctan\left(\frac{x}{2}\right)+c$ (b) $\frac{1}{2}\arctan(2x)+c$

(c) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (d) $\int \frac{1}{\sqrt{4-4x^2}}\,dx$

(c) $\arcsin(x)+c$ (d) $\frac{1}{2}\arcsin(x)+c$

(e) $\int sec^2(x)\,dx$ (f) $\int sec^2(2x)\,dx$ (g) $\int -csc^2(x)\,dx$

(e) $\tan(x)+c$ (f) $\frac{1}{2}\tan(2x)+c$ (g) $\cot(x)+c$

(2) Integrate the following:

(a) $\int sin^2(x)\,dx$

$$\int sin^2(x)\,dx = 0.5\int -cos2x+1dx$$

$$-0.25sin2x+0.5x+c$$

(b) $\int \sin(x)\cos(x)\,dx$

$$-\frac{1}{2}\cos^2(x)+c$$

(3) Integrate the following:

(a) $\int \frac{2x}{\sqrt{4+x^2}}\,dx$ (b) $\int \frac{1}{x(lnx)^2}\,dx$

(a) $2(4+x^2)^{0.5}+c$ (b) $-(lnx)^{-1}+c$

(4) By considering that $\frac{1}{1+e^x}=\frac{1+e^x-e^x}{1+e^x}$, find the integral of $\int \frac{1}{1+e^x}\,dx$

$$\int \frac{1}{1+e^x}\,dx=\int \frac{1+e^x-e^x}{1+e^x}\,dx$$

$$\int 1+\frac{-e^x}{1+e^x}\,dx$$

$$x+-\ln(1+e^x)+c$$

(5) Use the substitution $u=2x-1$ to find the integral $\int 2x(2x-1)^3\,dx$

$$\frac{du}{dx}=2$$

$$x=\frac{u+1}{2}$$

$$\int 2x(2x-1)^3\,dx=\int 2\frac{u+1}{2}u^3\frac{du}{2}$$

$$\frac{1}{2}\int u^4+u^3du=\frac{1}{2}\left(\frac{u^5}{5}+\frac{u^4}{4}\right)+c$$

$$\frac{1}{2}\left(\frac{(2x-1)^5}{5}+\frac{(2x-1)^4}{4}\right)+c$$

(6) By using the substitution $x = a\tan\theta$ show that:

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$x = a\tan\theta$$

$$\theta = \arctan\left(\frac{x}{a}\right)$$

$$dx = a\sec^2\theta d\theta$$

$$\int \frac{1}{a^2 + (a\tan\theta)^2}\, a\sec^2\theta d\theta$$

$$\int \frac{1}{a^2\sec^2\theta}\, a\sec^2\theta d\theta = \frac{1}{a}\int 1 d\theta$$

$$\frac{1}{a}\theta + c$$

$$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

(7) By using the substitution $x = sin\theta$ show that:

$$\int \sqrt{1-x^2}dx = \int cos^2(\theta)d\theta$$

$$x = sin\theta$$

$$dx = cos\theta d\theta$$

$$\int \sqrt{1-x^2}dx = \sqrt{1-(sin\theta)^2}cos\theta d\theta$$

$$\int cos^2\theta d\theta$$

(b) Hence find $\int_0^{0.5} \sqrt{1-x^2}dx$

$$\int \cos^2(x)\,dx = 0.5\int \cos(2x) + 1\,dx$$

$$= 0.5(0.5sin(2x) + x) + c$$

$$= \frac{sin(2x) + 2x}{4} + c$$

$$\int_0^{0.5} \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{6}} cos^2\theta d\theta$$

$$\int_0^{\frac{\pi}{6}} cos^2\theta d\theta = \left[\frac{sin(2x) + 2x}{4}\right]_0^{\frac{\pi}{6}}$$

$$\frac{\sqrt{3}}{8} + \frac{\pi}{12}$$

(8) By first writing $\frac{2x+1}{x^2+x-2}$ as the sum of 2 fractions with linear denominators, calculate the following:

$$\int_2^3 \frac{2x+1}{x^2+x-2}\,dx$$

Leave your answer in the form $\ln\left(\frac{a}{b}\right)$ where $a, b \in \mathbb{Z}$.

$$\frac{2x+1}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$\frac{2x+1}{x^2+x-2} = \frac{1}{x+2} + \frac{1}{x-1}$$

$$\int_2^3 \frac{2x+1}{x^2+x-2}\,dx = \int_2^3 \frac{1}{x+2} + \frac{1}{x-1}\,dx$$

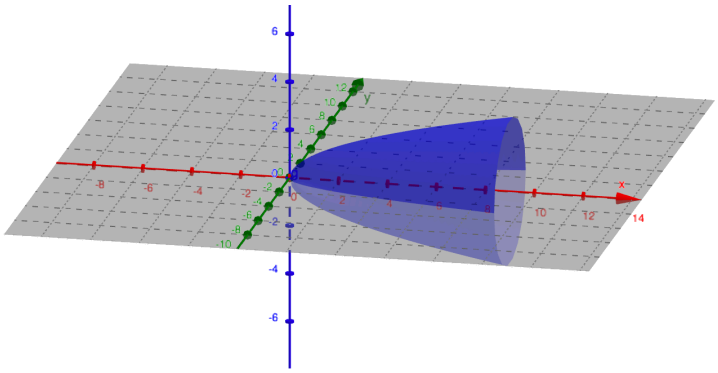
$$[\ln(x+2) + \ln(x-1)]_2^3$$

$$ln5 + ln2 - (ln4 + ln1) = ln\left(\frac{5}{2}\right)$$

Volume of revolution

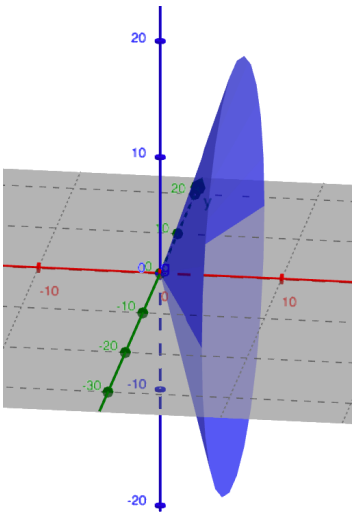
Name……

(1) Find the volume of revolution when the line $y = \sqrt{x},\ 0 \leq x \leq 9$ is rotated 360° around the x-axis.



$$\pi \int_0^9 x\,dx = \frac{81}{2}\pi$$

(2) Find the volume of revolution when the line $y = 3x\ \ 0 \leq x \leq 6$ is rotated 360° around the x-axis.



$$\pi \int_0^6 9x^2\,dx = 648\pi$$

(b) Show that you can achieve the same result by considering the volume of the cone shown above.

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2 h$$

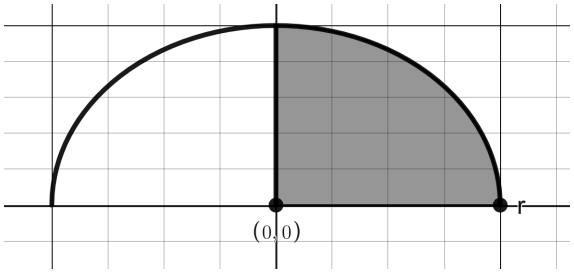
$$h = 6$$

$$r = 18$$

$$V = \frac{1}{3}\pi 18^2(6) = 648\pi$$

(3) The general equation of a quarter-circle centred at (0,0) with radius r is:

$$y = \sqrt{r^2 - x^2} \quad 0 \le x \le r$$



Show by use of an appropriate volume of revolution method that the volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

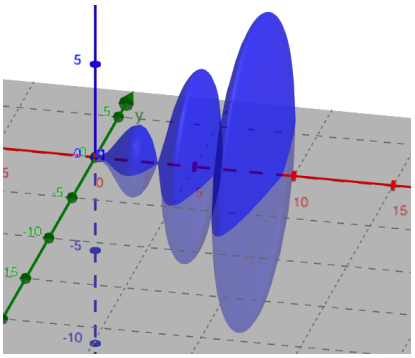
$$V = 2\pi \int_0^r r^2 - x^2 \, dx$$

$$V = 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$V = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3}\pi r^3$$

(4) [GDC REQUIRED] Find the volume of revolution when the curve shown below is rotated 360° around the x-axis.

$$y = x\sin(x), \qquad 0 \le x \le 3\pi$$



$$V = \pi \int_0^{3\pi} (x\sin(x))^2 \, dx = 431$$

(5) The graph of $y = \sqrt{x+1}$ between $x = 0$ and $x = a$ is rotated 360° around the x-axis. The volume of the solid formed is $\frac{3\pi}{2}$. Find a.

$$V = \pi \int_0^a x + 1 \, dx = \frac{3\pi}{2}$$

$$V = \pi \left[\frac{x^2}{2} + x\right]_0^a = \pi\left(\frac{a^2}{2} + a\right)$$

$$\frac{a^2}{2} + a = \frac{3}{2}$$

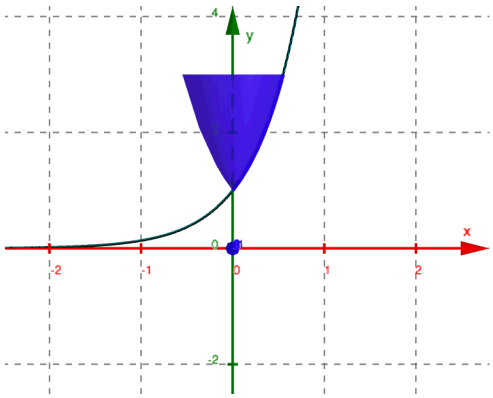
$$a^2 + 2a - 3 = 0$$

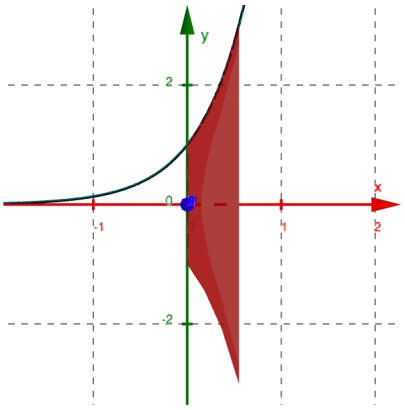
$$a = 1$$

Volume of revolution and kinematics HL

Name……

(1) For this question we investigate volumes formed by the curve $f(x) = e^{2x}$ $1 \leq y \leq 3$.





(a) A bowl is formed by rotating the curve $f(x)$ about the x-axis by 360 degrees. Find the volume of the bowl formed.

$$\pi \int_0^{0.5ln3} e^{4x}\, dx = 2\pi$$

(b) A different bowl is formed by rotating the curve $f(x)$ about the y-axis by 360 degrees. What is the percentage decrease in the new bowl's volume compared with the volume in (a)?

$$\pi \int_1^3 \left(\frac{\ln y}{2}\right)^2 dy = 0.8083107022$$

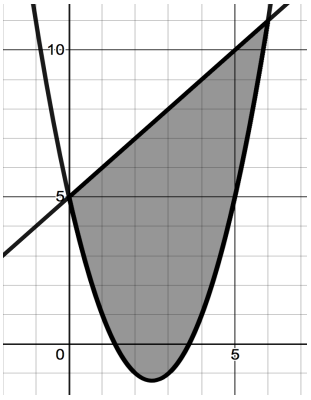
$$\%\ change = \frac{2\pi - 0.8083107022}{2\pi} \times 100 = 87.1\%$$

(c) Find k so that when the curve $y = e^{2x}$ $1 \leq y \leq k$ is rotated around the y-axis the bowl has a volume of 1 unit cubed.

$$\pi \int_1^k \left(\frac{\ln y}{2}\right)^2 dy = 1$$

$$k = 3.19$$

(2) Find the volume of revolution when the area enclosed between the line $y = x + 5$ and the curve $y = x^2 - 5x + 5$ is rotated 360 degrees about the x-axis.



$$x^2 - 5x + 5 = x + 5$$

$$x = 0, 6$$

$$\pi \int_0^6 (x + 5)^2 \, dx - \pi \int_0^6 (x^2 - 5x + 5)^2 \, dx = 995$$

(3) A particle starts at the origin and moves along a straight line so that its displacement in metres after t seconds is given by:

$$s^2 + s + t = 0$$

(a) Show that the velocity of the particle can be written as:

$$\frac{ds}{dt} = -\frac{1}{2s + 1}$$

$$2s\frac{ds}{dt} + \frac{ds}{dt} + 1 = 0$$

$$\frac{ds}{dt}(2s + 1) = -1$$

$$\frac{ds}{dt} = -\frac{1}{2s + 1}$$

(b) Find the initial velocity of the particle.

$$v = -\frac{1}{2s+1}$$

When t =0, s = 0. Therefore

$$v(0) = -\frac{1}{2(0)+1} = -1$$

(c) Find an equation for the acceleration of the particle in terms of s.

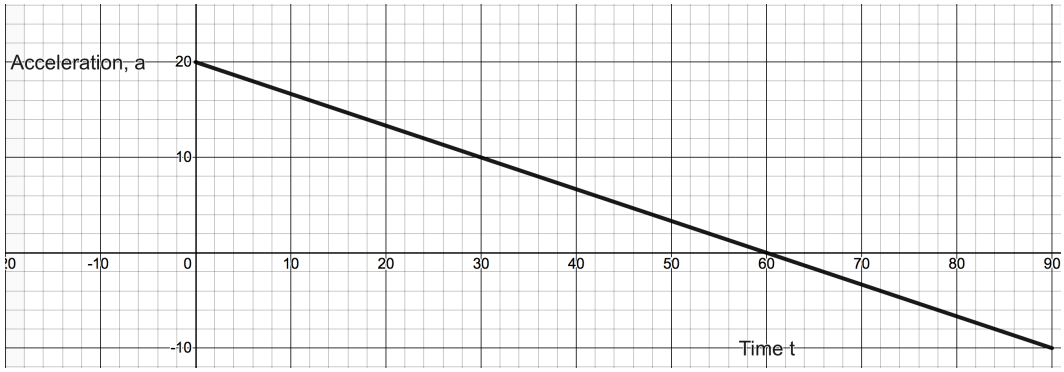
$$v = -(2s+1)^{-1}$$

$$\frac{dv}{dt} = 2(2s+1)^{-2}\frac{ds}{dt}$$

$$a = 2(2s+1)^{-2}\frac{-1}{2s+1}$$

$$a = \frac{-2}{(2s+1)^3}$$

(4) Part of the acceleration $(a), ms^{-2}$ of a fighter jet travelling in a straight line over time $(t), seconds$ is shown below:



(a) Write down when the velocity of the fighter jet is a maximum. Explain your answer with reference to the graph.

Stationary point when t = 60 as a = 0. Acceleration is the gradient of the velocity graph. Acceleration changes from positive to negative therefore maximum.

(b) Find an equation for the acceleration of the fighter jet.

$$a = -\frac{30}{90}t + 20$$

$$a = -\frac{1}{3}t + 20$$

(c) The fighter jet is initially travelling at a velocity of 200 ms^{-1}. Find for how many seconds it is travelling faster than Mach 1 (340 ms^{-1}) for $0 \leq t \leq 120$.

$$v = -\frac{1}{6}t^2 + 20t + c$$

$$v = -\frac{1}{6}t^2 + 20t + 200$$

$$v = -\frac{1}{6}t^2 + 20t + 200 > 340$$

intersection: $t = 7.46, t = 112.5$

therefore 105 seconds.

(c) Find the total distance travelled by the fighter jet over the first 90 seconds.

$$\int_0^{90} \left| -\frac{1}{6}t^2 + 20t + 200 \right| dt = 58500$$

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Continuous probability density functions

Name……

(1) The continuous random variable X has probability density function that is defined as:

$$f(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

(a) Explain why this meets the conditions of a probability density function.

$$\int_0^1 4x^3 dx = 1.$$

$f(x) \geq 0$ for all x in the domain.

(b) Write down the mode of X.

Mode when $f(x)$ greatest. Therefore when $x = 1$.

(c) Find $E(X)$

$$\int_0^1 4x^4 dx = \left[\frac{4x^5}{5}\right]_0^1 = \frac{4}{5}$$

(ii) Find $Var(X)$

$$\int_0^1 4x^5 dx - \left(\frac{4}{5}\right)^2$$

$$\left[\frac{4x^6}{6}\right]_0^1 - \frac{16}{25} = \frac{2}{75}$$

(iii) Find the median of X.

$$\int_0^a 4x^3 dx = 0.5.$$

$$[x^4]_0^a = 0.5$$

$$a = \sqrt[4]{\frac{1}{2}}$$

(d) This number generator is used to determine when someone wins on a slot machine game. If the number generator is set to pay out when $0 \leq x \leq 0.25$ find the probability that a player wins.

$$\int_0^{\frac{1}{4}} 4x^3 dx = \frac{1}{256}$$

(e) If the slot machine costs \$1 to play and pays out \$11 what is the expected profit or loss for a player on this machine?

$$\frac{1}{256}(+10) + \frac{255}{256}(-1) = -\frac{245}{256}$$

(f) In order for this to be a fair machine the operator sets the machine to pay out when $0 \leq x \leq k$. Find k.

$$\int_0^k 4x^3 dx = k^4$$

$$k^4(+10) + (1 - k^4)(-1) = 0$$

$$11k^4 - 1 = 0$$

$$k = \sqrt[4]{\frac{1}{11}}$$

(2) [GDC] The continuous random variable X has probability density function that is defined as:

$$f(x) = \begin{cases} \frac{k}{x^2+3} & 0 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$$

(a) Find k. Give your answer to 4 significant figures.

$$k\int_0^2 \frac{1}{x^2+3} = 1$$

$$k = 2.020893126\ldots$$

$$k = 2.021$$

(b) Find the median of X.

$$\int_0^a \frac{2.020893126\ldots}{x^2+3} = 0.5$$

$$a = 0.791$$

(3) [GDC]The continuous random variable X has probability density function that is defined as:

$$f(x) = \begin{cases} \frac{2\cos^2(x)}{\pi} & 0 \le x \le \pi \\ 0 & elsewhere \end{cases}$$

(a) Find $E(X)$

$$\int_0^{\pi} x\,\frac{2\cos^2(x)}{\pi}\,dx = \frac{\pi}{2}$$

(b) Find $p(X > E(X))$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{2\cos^2(x)}{\pi}\,dx = \frac{1}{2}$$

(c) Given that $X > \frac{\pi}{2}$ find $p\left(X < \frac{3\pi}{4}\right)$.

$$\frac{p\left(\frac{\pi}{2} < X < \frac{3\pi}{4}\right)}{p\left(X > \frac{\pi}{2}\right)} = \frac{\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{2\cos^2(x)}{\pi}\,dx}{\frac{1}{2}} = \frac{0.090845\ldots}{0.5}$$

$$= 0.182$$